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ADAPTIVE REGULATION FOR DETERMINISTIC SYSTEMS*†‡

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Abstract

For the linear deterministic system with unknown orders and coefficients adaptive controls are given so that the closed-loop system is stabilized and the unknown parameters are consistently estimated. Moreover, if the parameter estimation is ignored, then the system input and output can be reduced to zero with an exponential rate.

1. Introduction

Let the SISO system be described by

$$A(z)y_n = B(z)u_n, \tag{1.1}$$

where u_n, y_n are the system input and output respectively, z is the shift-back operator and $A(z)$ and $B(z)$ are coprime polynomials:

$$A(z) = 1 + a_1 z + \cdots + a_{p_0} z^{p_0}, \quad a_{p_0} \neq 0, \ p_0 \geq 0, \tag{1.2}$$

$$B(z) = b_1 z + \cdots + b_{q_0} z^{q_0}, \quad b_{q_0} \neq 0, \ q_0 \geq 1. \tag{1.3}$$

The system coefficients

$$\theta = [-a_1 \cdots -a_{p_0} \quad b_1 \cdots b_{q_0}]^\tau \tag{1.4}$$

and the system orders (p_0, q_0) are unknown. It is assumed that a set containing the true orders (p_0, q_0) is known, i.e. $p^* \geq 1$ and $q^* \geq 1$ are given so that

$$(p_0, q_0) \in M \triangleq \{(p,q): \ 0 \leq p < p^*, \ 1 \leq q < q^*\}.$$

The problem discussed in this paper is that based on the observed data one wants to design adaptive control, that leads the output and input of the closed-loop system tending to

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zero, and simultaneously wants to consistently estimate the unknown orders and coefficients. This problem has been the research topic of a series papers[1-6], which can be classified into two groups: one devotes effort to controlling the system only, while the other one cares for both the control performance and consistency of the parameter estimation. Among the above-mentioned papers, [2,4,5,6] belong to the first group, and [1,3] to the second group. We note that all these papers need some extra conditions in addition to the standard coprimeness assumption. For example, in [1] it is assumed that p_0 is known and $z^{-1}B(z)$ is stable; in [2] $\max(p_0, q_0)$ is known; in [4,5] it is required that the true θ and the parameters in controller are located in a known region.

In this paper imposing no additional condition on $A(z)$ and $B(z)$ except coprimeness, we propose an adaptive regulator which controls the system output and input asymptotically approaching to zero and makes the estimates for coefficient and orders strongly consistent. The convergence rate of the coefficient estimate is also indicated. If the parameter estimation is ignored, then the system can be adaptively stabilized with an exponential rate.

It is worth noting that there is the essential difference for adaptive stabilization between two cases: 1) both p_0 and q_0 are unknown, and 2) either p_0 or q_0 is known. In the case 2), say, when p_0 is known, we may take $u_n = v_n$, $\forall n \geq 0$, where $\{v_n\}$ is a sequence of mutually independent random variables with

$$Ev_n^2 = \frac{1}{n^\varepsilon}, \quad v_n^2 \leq \frac{\sigma^2}{n^\varepsilon}, \quad \varepsilon \in \left(0, \frac{1}{2}\right), \quad \sigma > 0. \tag{1.5}$$

Similiar to the proof of Theorem 3 in [7] it can be shown

$$\liminf_{n\to\infty} \frac{1}{n^{1-\varepsilon}} \lambda_{\min}\left(\sum_{j=0}^{n} \varphi_j \varphi_j^\tau\right) \triangleq c > 0 \quad a.s.,$$

where and hereafter $\lambda_{\min}(X)$ denotes the minimum eigenvalue of a matrix X,

$$\varphi_n^\tau = [y_n \ \cdots \ y_{n-p_0+1} \quad u_n \ \cdots \ u_{n-q^*+1}]. \tag{1.6}$$

This means that for any fixed ω there is $n_0 < \infty$ such that

$$\det\left(\sum_{j=p_0+q^*}^{n-1} \varphi_j \varphi_j^\tau\right) > 0, \quad \forall n \geq n_0. \tag{1.7}$$

Therefore, the least squares estimate

$$\theta_n = \left(\sum_{j=p_0+q^*}^{n-1} \varphi_j \varphi_j^\tau\right)^{-1} \sum_{j=p_0+q^*}^{n-1} \varphi_j y_{j+1} \equiv \begin{bmatrix} \theta \\ 0 \end{bmatrix}, \quad \forall n \geq n_0 \tag{1.8}$$

exactly gives the true parameter starting from time n_0. Thus, one may proceed as follows: take $\{v_n\}$ as the system input and at each time verify whether or not (1.7) holds. If (1.7) is true for some n, then one simply obtains the system true parameter and may treat the problem as a non-adaptive one. The important thing is that this procedure terminates in a finite number of steps.

However, in the case 1), as will be shown in Lemma, $\sum_{j=p^*+q^*}^{n}$ is degenerate for any n if $p > p_0$ and $q > q_0$. One cannot say that $p = p_0$ even though

$$\det\left(\sum_{j=p^*+q^*}^{n}\varphi_j(p,q)\varphi_j^\tau(p,q)\right) > 0,$$
$$\det\left(\sum_{j=p^*+q^*}^{n}\varphi_j(p+1,q)\varphi_j^\tau(p+1,q)\right) = 0 \tag{1.9}$$

for many successive n, because it is not excluded that

$$\det\left(\sum_{j=p^*+q^*}^{n}\varphi_j(p+1,q)\varphi_j^\tau(p+1,q)\right) > 0$$

for some large n, where

$$\varphi_n^\tau(p,q) = [y_n \cdots y_{n-p+1} \quad u_n \cdots u_{n-q+1}]. \tag{1.10}$$

So one never knows if he has achieved the true θ or not.

This difficulty will be overcome in the sequel by choosing appropriate stopping times.

2. Main Results

Given initial value $\theta_0(p,q)$, let us define the estimate

$$\theta_n(p,q) = \left(I_{p+q} + \sum_{j=p^*+q^*}^{n-1}\varphi_j(p,q)\varphi_j^\tau(p,q)\right)^{-1}\sum_{j=p^*+q^*}^{n-1}\varphi_j(p,q)y_{j+1} \tag{2.1}$$

for the unknown coefficient

$$\theta(p,q) = [-a_1 \cdots -a_p \quad b_1 \cdots b_q]^\tau, \quad \forall (p,q)\in M, \tag{2.2}$$

where $a_i = 0$ for $i > p_0$, $b_j = 0$ for $j > q_0$ by definition and $\varphi_j(p,q)$ is given by (1.10).

It is well known that (2.1) can be written in a recursive form.

For order estimation[3] let us take a sequence $\{\mu_n\}$ of real numbers

$$\mu_n > 0, \quad \mu_n \to \infty \quad \text{and} \quad \frac{\mu_n}{n^{1-\varepsilon}} \to 0, \quad \varepsilon \in \left(0, \frac{1}{2}\right) \tag{2.3}$$

and set

$$\sigma_n(p,q) = \sum_{j=0}^{n-1}(y_{j+1} - \varphi_j^\tau(p,q)\theta_n(p,q))^2, \tag{2.4}$$

$$CIC(p,q)_n = \sigma_n(p,q) + (p+q)\mu_n. \tag{2.5}$$

The order estimate (p_n, q_n) is given by minimizing $CIC(p,q)_n$:

$$(p_n, q_n) = \operatorname*{argmin}_{(p,q)\in M} CIC(p,q)_n, \quad \forall n \geq 1, \tag{2.6}$$

while the coefficient $\theta(p_0, q_0)$ is estimated by (2.1) with $p = p_n$, $q = q_n$:

$$\theta_n(p_n, q_n) = [-a_{1n} \cdots - a_{p_n n} \quad b_{1n} \cdots b_{q_n n}]^\tau, \quad \forall n \geq 1. \tag{2.7}$$

We note that if $\{\mu_n\}$ satisfies (2.3), then $\{c\mu_n\}$ with any constant $c > 0$ also satisfies (2.3). It is clear that for finite n, (p_n, q_n) may vary with c, but as will be shown in Theorem 2 their limit does not depend on c. It is also clear that the constant c reflects the scale of $\{y_n\}$ and $\{u_n\}$.

We now define adaptive control. Set

$$A_n(z) = 1 + a_{1n}z + \cdots + a_{p_n n}z^{p_n}, \tag{2.8}$$

$$B_n(z) = b_{1n}z + \cdots + b_{q_n n}z^{q_n}, \tag{2.9}$$

$$r_n = \max\{|y_j|, \ |u_j|, \quad j = n - \max(p^*, q^*), \cdots, n-1\}. \tag{2.10}$$

For simplicity of notation we say that at time n "$\mathcal{A}$" holds if the equation

$$A_n(z)G_n(z) - B_n(z)H_n(z) = 1 \tag{2.11}$$

has a unique solution $(G_n(z), H_n(z))$ with

$$\deg(G_n(z)) \leq q_n - 1, \quad \deg(H_n(z)) \leq p_n - 1 \tag{2.12}$$

and

$$\|A_n(z)\| + \|B_n(z)\| + \|G_n(z)\| + \|H_n(z)\| \leq \varepsilon_n^{-1}, \tag{2.13}$$

and if

$$\|y_n - \varphi_{n-1}^\tau(p_n, q_n)\theta_n(p_n, q_n)\| \leq \varepsilon_n^2 r_n, \tag{2.14}$$

where $\{\varepsilon_n\}$ is an arbitrarily fixed sequence of real numbers with

$$\varepsilon_n \in \left(0, \frac{1}{2(p^* + q^*)}\right), \quad \varepsilon_n \to 0, \quad \varepsilon_n^2 \mu_n \to \infty \tag{2.15}$$

and by the norm of a polynomial $X(z) = \sum_{j=0}^{r} x_j z^j$ we mean $\|X(z)\| = \sum_{j=0}^{r} |x_j|$.

Let $\{\gamma_n\}$ be a sequence of positive real numbers, $\gamma_n \to 0$.

We say that at time n "$\mathcal{B}$" holds, if

$$\left(\sum_{j=p^*+q^*}^{n-1} \varphi_j(p_n, p^* + q^*)\varphi_j^\tau(p_n, p^* + q^*)\right) - \mu_n I > 0, \tag{2.16}$$

$$\lambda_{\min}\left(\sum_{j=p^*+q^*}^{n-1} \varphi_j(p_n + 1, q^*)\varphi_j^\tau(p_n + 1, q^*)\right) \leq \gamma_n, \tag{2.17}$$

$$\left(\sum_{j=p^*+q^*}^{n-1} \varphi_j(p^* + q^*, q_n)\varphi_j^\tau(p^* + q^*, q_n)\right) - \mu_n I > 0, \tag{2.18}$$

and

$$\lambda_{\min}\left(\sum_{j=p^*+q^*}^{n-1} \varphi_j(p^*, q_n + 1)\varphi_j^\tau(p^*, q_n + 1)\right) \leq \gamma_n. \tag{2.19}$$

Define adaptive control u_n as

$$u_n = \begin{cases} v_n, & \text{if } n \in [0, \tau_0) \text{ or } n \in [\tau_i, \sigma_{i+1}) \text{ for some } i \geq 0, \\ H_{\sigma_i}(z) y_n - (G_{\sigma_i}(z) - 1) u_n, & \text{if } n \in [\sigma_i, \tau_i) \text{ for some } i \geq 1, \end{cases} \tag{2.20}$$

where $\tau_0 = p^* + q^*$, $\{v_n\}$ is a sequence of mutually independent random variables with properties (1.5) and

$$\tau_0 < \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \cdots$$

are stopping times defined as follows:

$$\sigma_i = \min\{n : \ n > \tau_{i-1} \text{ and } \mathcal{A} \text{ and } \mathcal{B} \text{ hold at time } n\}, \tag{2.21}$$

$$\tau_i = \min\{n : \ n > \sigma_i \text{ and } |y_n - \varphi_{n-1}^\tau(p_{\sigma_i}, q_{\sigma_i}) \theta_{\sigma_i}(p_{\sigma_i}, q_{\sigma_i})| > \varepsilon_{\sigma_i}^2 r_n\}. \tag{2.22}$$

We note that $G_n(z)$ is a monic polynomial whenever $\mathcal{A}$ holds. Hence, u_n can be defined by (2.20) indeed.

It is easy to see that (2.11) and (2.12) are satisfied if and only if $\det M_n \neq 0$ where

$$M_n = [M_{1n} \quad M_{2n}] \tag{2.23}$$

with

$$M_{1n}^\tau = \overbrace{\begin{pmatrix} 1 & a_{1n} & \cdots & \cdots & \cdots & \cdots & a_{p_n n} & 0 & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 1 & a_{1n} & \cdots & \cdots & \cdots & \cdots & a_{p_n n} \end{pmatrix}}^{p_n + q_n} \Bigg\} q_n, \tag{2.24}$$

$$M_{2n}^\tau = \overbrace{\begin{pmatrix} 0 & -b_{1n} & \cdots & \cdots & \cdots & \cdots & -b_{q_n n} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & -b_{1n} & \cdots & \cdots & \cdots & \cdots & -b_{q_n n} \end{pmatrix}}^{p_n + q_n} \Bigg\} p_n. \tag{2.25}$$

In the case M_n is nondegenerate, the coefficients

$$\psi_n^\tau = [1 \ g_{1n} \cdots g_{q_n - 1n} \quad h_{0n} \cdots h_{p_n - 1n}]$$

of the polynomials

$$G_n(z) = 1 + \sum_{j=1}^{q_n - 1} g_{jn} z^j, \quad \text{and} \quad H_n(z) = \sum_{j=0}^{p_n - 1} h_{jn} z^j \tag{2.26}$$

are given by

$$\psi_n = M_n^{-1} e_n, \tag{2.27}$$

where

$$e_n^\tau = [1 \ 0 \cdots 0]_{1 \times (p_n + q_n)}.$$

Theorem 1. If $A(z)$ and $B(z)$ are coprime, then the input and output of the adaptive control system (1.1) and (2.20) exponentially tend to zero:

$$|y_n| + |u_n| \le A\lambda^n, \quad A > 0, \quad \lambda \in (0,1), \quad \forall n \ge 0. \tag{2.28}$$

This Theorem does not concern the parameter estimation problem which is considered in Theorem 2.

Let us take $\varepsilon \in \left(0, \frac{1}{2(p^*+q^*)}\right)$ instead of $\varepsilon \in \left(0, \frac{1}{2}\right)$ in (1.5) and (2.3) and disturb the control defined by (2.20). To be specific, we define

$$u_n = \begin{cases} v_n, & \text{if } n \in [0, \tau_0) \text{ or } n \in [\tau_i, \sigma_{i+1}) \text{ for some } i \ge 0, \\ H_{\sigma_i}(z)y_n - (G_{\sigma_i}(z) - 1)u_n + v_n, & \text{if } n \in [\sigma_i, \tau_i) \text{ for some } i \ge 1. \end{cases}$$

Theorem 2. If $A(z)$ and $B(z)$ are coprime, then the adaptive control (2.29) makes the system input and output asymptotically tending to zero and the estimates for orders and coefficients strongly consistent, namely,

$$|y_n| + |u_n| = O\left(\frac{1}{n^{\varepsilon/2}}\right), \tag{2.30}$$

$$\lim_{n\to\infty} (p_n, q_n) = (p_0, q_0), \tag{2.31}$$

$$\|\theta_n(p_n \vee p_0, q_n \vee q_0) - \theta(p_n \vee p_0, q_n \vee q_0)\| = O\left(\frac{1}{n^{\varepsilon(p^*+q^*)}}\right). \tag{2.32}$$

3. Lemmas

We first prove some lemmas.

Lemma 1. If $A(z)$ and $B(z)$ are coprime and if

$$\lim_{n\to\infty} (p_n, q_n) = (p_0, q_0), \tag{3.1}$$

$$\lim_{n\to\infty} \|\theta_n(p_n \vee p_0, q_n \vee q_0) - \theta(p_n \vee p_0, q_n \vee q_0)\| = 0, \tag{3.2}$$

then for sufficiently large n (2.11)–(2.13) are satisfied.

Proof. From coprimeness of $A(z)$ and $B(z)$ it follows that (2.11) and (2.12) hold with $A_n(z)$, $B_n(z)$, $H_n(z)$, $G_n(z)$, p_n and q_n replaced by $A(z)$, $B(z)$, $H(z)$, $G(z)$, p_0 and q_0 respectively. Then the conclusion of the lemma is derived immediately from consistency of (p_n, q_n) and $\theta_n(p_n, q_n)$ and the expression (2.27). ∎

Lemma 2. For $p \ge p_0 + 1$ and $q \ge q_0 + 1$ the matrix $A_n \triangleq \sum_{j=p^*+q^*}^{n} \varphi_j(p,q)\varphi_j^\tau(p,q)$ is degenerate, $\forall n \ge p^* + q^*$, i.e.,

$$\lambda_{\min}(A_n) = 0, \quad \forall n \ge p^* + q^*. \tag{3.3}$$

Proof. By (1.1) it is easy to see that $\alpha^\tau \varphi_j(p,q) = 0$ with

$$\alpha^\tau = [1\ a_1\ \cdots\ a_{p_0}\ \overbrace{0\ \cdots\ 0}^{p-p_0-1}\ 0\ -b_1\ \cdots\ -b_{q_0}\ \overbrace{0\ \cdots\ 0}^{q-q_0-1}].$$

∎

Lemma 3. Let $\{s_n\}$ be generated by

$$s_{n+2i} = \alpha_{2i-1}s_{n+2i-1} + \cdots + \alpha_0 s_n + \beta_n, \quad \forall n \geq 0 \tag{3.4}$$

with $\sum_{j=0}^{2i-1} |\alpha_j| < 1$. If $|\beta_n| = O(n^{-\alpha})$, $\alpha \in (0,1)$, then $s_n = O(n^{-\alpha})$, and if $\beta_n \equiv 0$, then $|s_n| \leq c\lambda^n$, $\forall n \geq 1$, $c > 0$, $\lambda \in (0,1)$.

Proof. Set

$$B = \begin{bmatrix} \alpha_{2i-1} & \cdots & \cdots & \cdots & \alpha_0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad Z_n = \begin{bmatrix} s_{n+2i-1} \\ \vdots \\ \vdots \\ s_n \end{bmatrix}, \quad e = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2i\times 1}$$

From (3.4) it is easy to verify that

$$Z_{n+1} = BZ_n + \beta_n e = B^{n+1}Z_0 + \sum_{i=0}^{n} \beta_i B^{n-i} e.$$

Noticing that

$$\det(zI - B) = z^{2i} - \alpha_{2i-1}z^{2i-1} - \cdots - \alpha_0$$

and

$$\begin{aligned} &\left| z^{2i} - \alpha_{2i-1}z^{2i-1} - \cdots - \alpha_0 \right| \Big|_{|z|\geq 1} \\ &\geq |z|^{2i}\left(1 - \sum_{j=0}^{2i-1} |\alpha_j||z|^{-(2i-j)}\right)\Big|_{|z|\geq 1} \\ &\geq 1 - \sum_{j=0}^{2i-1} |\alpha_j| > 0, \end{aligned}$$

we find that all eigenvalues of B are in the open unit disk. Therefore,

$$\|B^n\| \leq c_1\lambda^n, \quad \forall n \geq 0 \quad \text{for some} \quad \lambda \in (0,1), \quad c_1 > 0 \tag{3.6}$$

and

$$|s_n| \leq \|Z_{n+1}\| \leq O(\lambda^{n+1}) + O\left(\sum_{i=1}^{n} i^{-\alpha}\lambda^{\frac{n-i}{2}}\lambda^{\frac{n-i}{2}}\right).$$

Since $\left(\frac{i}{n}\right)^{-\alpha}\lambda^{\frac{n-i}{2}}$ is uniformly bounded in i and n, $1 \leq i \leq n$, we have

$$|s_n| \leq \|Z_n\| = O(n^{-\alpha})$$

in the case $|\beta_n| = O(n^{-\alpha})$. The second conclusion of the lemma immediately follows from (3.5) and (3.6). ∎

Lemma 4. In both cases of Theorems 1 and 2 there exists $i \geq 0$ such that $\sigma_i < \infty$ and $\tau_i = \infty$.

Proof. If $\tau_i < \infty$, $\sigma_{i+1} = \infty$ for some $i \geq 0$, then by an argument similar to that used in the proof of Theorem 3 of [7] we see that

$$\frac{1}{n^{1-\epsilon}}\left(\sum_{j=p^*+q^*}^{n-1} \varphi_j(p_0, p^*+q^*)\varphi_j^\tau(p_0, p^*+q^*)\right) \geq cI > 0, \quad \forall n \geq n_0 \tag{3.7}$$

and

$$\frac{1}{n^{1-\epsilon}}\left(\sum_{j=p^*+q^*}^{n-1} \varphi_j(p^*+q^*, q_0)\varphi_j^\tau(p^*+q^*, q_0)\right) \geq cI > 0, \quad \forall n \geq n_0, \tag{3.8}$$

where n_0 is sufficiently large, and n_0 may depend on sample.

By Theorem 3.2 of [3], (3.7) and (3.8) imply that $(p_n, q_n) \underset{n\to\infty}{\longrightarrow} (p_0, q_0)$. Therefore, there exists some $n_1 \geq n_0$ such that

$$(p_n, q_n) \equiv (p_0, q_0), \quad \forall n \geq n_1. \tag{3.9}$$

Lemma 2 together with (3.7)–(3.9) means that "$\mathcal{B}$" holds $\forall n \geq n_1$. From (2.1), (3.8), (3.9) and (2.3) it follows that $\forall n \geq n_1$,

$$\|\theta_n(p_n, q_n) - \theta\| = O\left(\frac{1}{n^{1-\epsilon}}\right), \tag{3.10}$$

which yields

$$\begin{aligned}&|y_n - \varphi_{n-1}^\tau(p_n, q_n)\theta_n(p_n, q_n)| = |\varphi_{n-1}^\tau(p_n, q_n)(\theta - \theta_n(p_n, q_n))|\\ =&|\varphi_{n-1}^\tau(p_0, q_0)(\theta - \theta_n(p_0, q_0))| \leq (p_0 + q_0)r_n\|\theta - \theta_n(p_0, q_0)\|\\ =&O\left(\frac{r_n}{n^{1-\epsilon}}\right).\end{aligned}$$

This incorporating (3.9), (3.10) and Lemma 1 yields that "$\mathcal{A}$" holds for all sufficiently large n. Thus, σ_{i+1} must be finite.

The obtained contradiction shows that "$\tau_i < \infty$, $\sigma_{i+1} = \infty$" is impossible.

Now, let $\sigma_i < \tau_i < \infty$, $\forall\, i \geq 0$.

Since $(p_{\sigma_i}, q_{\sigma_i}) \in M$, there exists a convergent subsequence, which is also denoted by $(p_{\sigma_i}, q_{\sigma_i})$ for notational simplicity but without loss of generality. Let (p', q') be the limit of the subsequence. Being the integers, $(p_{\sigma_i}, q_{\sigma_i}) \equiv (p', q')$ for i starting from some i_0.

By the definition of (2.21) we have

$$\left(\sum_{j=p^*+q^*}^{\sigma_i-1} \varphi_j(p', p^*+q^*)\,\varphi_j^\tau(p', p^*+q^*)\right) - \mu_{\sigma_i} I > 0, \quad \forall i \geq i_0, \tag{3.11}$$

$$\lambda_{\min}\left(\sum_{j=p^*+q^*}^{\sigma_i-1} \varphi_j(p'+1, q^*)\,\varphi_j^\tau(p'+1, q^*)\right) \leq \gamma_{\sigma_i}, \quad \forall i \geq i_0, \tag{3.12}$$

$$\left(\sum_{j=p^*+q^*}^{\sigma_i-1} \varphi_j(p^*+q^*, q')\,\varphi_j^\tau(p^*+q^*, q')\right) - \mu_{\sigma_i} I > 0, \quad \forall i \geq i_0 \tag{3.13}$$

and

$$\lambda_{\min}\left(\sum_{j=p^*+q^*}^{\sigma_i-1}\varphi_j(p^*,q'+1)\,\varphi_j^\tau(p^*,q'+1)\right)\le\gamma_{\sigma_i},\quad \forall i\ge i_0. \tag{3.14}$$

We now show $p'=p_0$.

From (3.12) it is clear that there is a sequence of unit vector η_i such that

$$|\eta_i^\tau\varphi_j(p'+1,q^*)|\le\sqrt{\gamma_{\sigma_i}},\quad \forall j\in[0,\sigma_i). \tag{3.15}$$

Let $\{\eta_{i_k}\}$ be a convergent subsequence of $\{\eta_i\}$: $\lim_{k\to\infty}\eta_{i_k}=\eta$, $\|\eta\|=1$.

For any fixed $j\ge p^*+q^*$ and any $i_k>j$ from (3.15) it follows that

$$\begin{aligned}0\le&|\eta^\tau\varphi_j(p'+1,q^*)|\\ \le&|\eta_{i_k}^\tau\varphi_j(p'+1,q^*)|+\|\eta-\eta_{i_k}^\tau\|\,\|\varphi_j(p'+1,q^*)\|\xrightarrow[k\to\infty]{}0\end{aligned}$$

or

$$\eta^\tau\varphi_j(p'+1,q^*)=0,\quad \forall j\ge p^*+q^*. \tag{3.16}$$

Let

$$\eta^\tau=[\alpha_0\cdots\alpha_{p'}\quad\beta_0\cdots\beta_{q*-1}]. \tag{3.17}$$

We note that $\alpha_{p'}$ must differ from 0, because otherwise we would have

$$\xi^\tau\varphi_j(p',p^*+q^*)=0,\quad \forall j\ge p^*+q^*$$

with

$$\xi^\tau=[\alpha_0\cdots\alpha_{p'-1}\quad\beta_0\cdots\beta_{q*-1}\overbrace{0\cdots0}^{p^*}],$$

which contradicts (3.11).

Set

$$D(z)=\sum_{j=0}^{p'}\alpha_jz^j,\quad E(z)=-\sum_{j=0}^{q^*-1}\beta_jz^j. \tag{3.18}$$

By (3.16) we have

$$D(z)y_n=E(z)u_n,\quad \forall n\ge p^*+q^*. \tag{3.19}$$

If $p'<p_0$, then there exists a polynomial $F(z)$ of degree γ with $\gamma<p'$ such that

$$A(z)=M(z)D(z)+F(z), \tag{3.20}$$

where $M(z)$ is a polynomial of degree p_0-p'. From (1.1) subtracting $M(z)D(z)y_n$, which equals $M(z)E(z)u_n$ by (3.19), we derive

$$F(z)y_n=(B(z)-M(z)E(z))u_n,\quad \forall n\ge p^*+q^*. \tag{3.21}$$

However, from (3.11) it follows that

$$\lambda_{\min}\left(\sum_{j=p^*+q^*}^{n}\varphi_j(s,p^*+q^*)\,\varphi_j^\tau(s,p^*+q^*)\right)\xrightarrow[n\to\infty]{}\infty,\quad \forall s\le p',$$

which means that (3.21) is possible only if $F(z) \equiv 0$ and $B(z) \equiv M(z)E(z)$. This together with (3.20) implies that $A(z)$ and $B(z)$ have a common factor $M(z)$, which must have zero degree because $A(z)$ and $B(z)$ are coprime by assumption. Hence $p' = p_0$.

In a completely similar way, from (3.13) and (3.14) we find that $q' = q_0$. Therefore,

$$(p_{\sigma_i}, q_{\sigma_i}) \equiv (p_0, q_0), \quad \forall i \geq i_0 \tag{3.22}$$

and from (2.1) and (2.16) we have

$$\|\theta_{\sigma_i}(p_0, q_0) - \theta(p_0, q_0)\| = O\left(\frac{1}{\mu_{\sigma_i}}\right), \quad \forall i \geq i_0. \tag{3.23}$$

Hence, by (2.15) there exists $i_1 \geq i_0$ such that

$$\begin{aligned} &|y_n - \varphi^{\tau}_{n-1}(p_0, q_0)\theta_{\sigma_i}(p_0, q_0)| \\ \leq &(p_0 + q_0) r_n \|\theta(p_0, q_0) - \theta_{\sigma_i}(p_0, q_0)\| \\ = &\left(\frac{r_n}{\mu_{\sigma_i}}\right) \leq \varepsilon^2_{\sigma_i} r_n, \quad \forall n \geq \sigma_{i_1}, \end{aligned} \tag{3.24}$$

which means $\tau_{i_1} = \infty$, a contradiction to $\tau_i < \infty$ for all i.

Therefore, the only possible case is "$\sigma_i < \infty$, $\tau_i = \infty$" for some i. ∎

4. Proof of Theorems

We are now in a position to prove our theorems.

Proof of Theorem 1.

By Lemma 4 we have $\sigma_i < \infty$, $\tau_i = \infty$ for some i. Noticing

$$A_{\sigma_i}(z)G_{\sigma_i}(z) - B_{\sigma_i}(z)H_{\sigma_i}(z) = 1 \tag{4.1}$$

and by (2.20) we see that for any $n \geq n_0 \triangleq \sigma_i + \max(p^*, q^*)$,

$$\begin{aligned} y_n &= A_{\sigma_i}(z)G_{\sigma_i}(z)y_n - B_{\sigma_i}(z)H_{\sigma_i}(z)y_n \\ &= G_{\sigma_i}(z)[A_{\sigma_i}(z)y_n - B_{\sigma_i}(z)u_n] + B_{\sigma_i}(z)[G_{\sigma_i}(z)u_n - H_{\sigma_i}(z)y_n] \\ &= G_{\sigma_i}(z)[A_{\sigma_i}(z)y_n - B_{\sigma_i}(z)u_n] \end{aligned} \tag{4.2}$$

and, similarly,

$$u_n = H_{\sigma_i}(z)[A_{\sigma_i}(z)y_n - B_{\sigma_i}(z)u_n]. \tag{4.3}$$

Paying attention to (2.13) and (2.22) from (4.2) and (4.3) we see that

$$\begin{aligned} |y_n| \leq &\|G_{\sigma_i}(z)\| \sum_{j=0}^{q^*-1} |y_{n-j} - \varphi^{\tau}_{n-j-1}(p_{\sigma_i}, q_{\sigma_i})\theta_{\sigma_i}(p_{\sigma_i}, q_{\sigma_i})| \\ \leq &\varepsilon^{-1}_{\sigma_i}\varepsilon^2_{\sigma_i} \sum_{j=0}^{q^*-1} r_{n-j} \leq \frac{1}{2(p^*+q^*)} \sum_{j=0}^{q^*-1} r_{n-j} \end{aligned}$$

and

$$|u_n| \leq \frac{1}{2(p^*+q^*)} \sum_{j=0}^{p^*-1} r_{n-j}. \tag{4.5}$$

Hence, we have

$$r_{n+2\mu} \leq \frac{1}{2(p^*+q^*)} \sum_{j=1}^{2\mu-1} r_{n+2\mu-j}, \tag{4.6}$$

where $\mu = \max(p^*, q^*)$.

Identify s_n with r_n for $n = n_0, n_0+1, \cdots, n_0+2\mu-1$ and take

$$s_{n+2\mu} = \frac{1}{2(p^*+q^*)} \sum_{j=1}^{2\mu-1} s_{n+2\mu-j}, \quad \text{for } n \geq n_0. \tag{4.7}$$

Then we have $s_n \geq r_n \geq 0$, $\forall n \geq n_0$, and by Lemma 3, (2.28) follows. ∎

Proof of Theorem 2.

By Lemma 4 we have $\sigma_i < \infty$, $\tau_i = \infty$ for some i.

We note that (2.29) differs from (2.20) by a term v_n, so corresponding to (4.2) and (4.3) we now have, for any $n \geq n_0 \stackrel{\Delta}{=} \sigma_i + \max(p^*, q^*)$,

$$y_n = G_{\sigma_i}(z)[A_{\sigma_i}(z)y_n - B_{\sigma_i}(z)u_n] + B_{\sigma_i}(z)v_n$$

and

$$u_n = H_{\sigma_i}(z)[A_{\sigma_i}(z)y_n - B_{\sigma_i}(z)u_n] + A_{\sigma_i}(z)v_n.$$

In the present case (4.4) and (4.5) change to

$$|y_n| \leq \|G_{\sigma_i}(z)\| \sum_{j=0}^{q^*-1} \left|y_{n-j} - \varphi^\tau_{n-j-1}(p_{\sigma_i}, q_{\sigma_i})\theta_{\sigma_i}(p_{\sigma_i}, q_{\sigma_i})\right| + \|B_{\sigma_i}\| \frac{q^*\sigma^2}{(n-q^*)^{\epsilon/2}}$$

$$\leq \frac{1}{2(p^*+q^*)} \sum_{j=0}^{q^*-1} r_{n-j} + \frac{c}{(n-q^*)^{\epsilon/2}}$$

and

$$|u_n| \leq \frac{1}{2(p^*+q^*)} \sum_{j=0}^{p^*-1} r_{n-j} + \frac{c}{(n-p^*)^{\epsilon/2}},$$

where $c = \frac{\max(p^*,q^*)\sigma}{\epsilon_{\sigma_i}}$.

Applying Lemma 3 leads to (2.30).

Let us denote $u_n^0 = u_n - v_n$. Then by (1.5) and (2.23) it is easy to see that $\frac{1}{n}\sum_{i=0}^n (u_i^0)^2 = O(n^{-\epsilon/2})$.

By Theorem 3.4 of [3] we have (2.31) and (2.32). ∎

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